

Lifshitz Hydrodynamics

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We construct the hydrodynamics of quantum field theories with a Lifshitz scaling symmetry. New transport coefficients are allowed by the absence of boost invariance, however, only one is compatible with a local increase of the entropy density. The formulation is applicable, in general, to fluids with an explicit breaking of boost symmetry. We use a Drude model of a strange metal to study the physical effects of the new transport coefficient. It can be measured using electric fields with non-zero gradients, or via the heat production when an external force is turned on. Scaling arguments fix the resistivity to be linear in the temperature.

Introduction.- Quantum critical points [1, 2] are believed to be responsible for the strange metal behaviour observed in a variety of materials including heavy-fermion compounds and high T_c superconductors. Although the quantum critical point is strictly defined at zero temperature, its properties affect a large region of the phase diagram.

As in ordinary critical points [3], there is a scaling symmetry at the critical point acting on the space-time coordinates as

$$t \rightarrow \Omega^z t, \quad x^i \rightarrow \Omega x^i, \quad i = 1, \dots, d. \quad (1)$$

We assume that at the relevant length scales the system is isotropic and homogeneous, thus rotation and translations (both in space and time) are symmetries. All together this is the ‘Lifshitz symmetry group’, inspired by the scaling in Lifshitz points [4, 5].

We use symmetries and the laws of thermodynamics as guiding principles in order to write the hydrodynamic equations of quantum critical points at finite temperature. We expect to have a universal expansion in derivatives of the velocity and temperature with some coefficients, whose precise values depend on the microscopic theory. Fluid dynamics have been shown in many cases to describe strongly correlated critical systems, such as conformal field theories at finite temperature [6], fermions at unitarity [7] and graphene [8–10].

An important aspect of theories with Lifshitz scaling symmetry is the absence of boost invariance. The energy-momentum tensor, which is symmetric for theories with Lorentz invariance, can receive now asymmetric contributions, and in principle many new transport coefficients can appear. We use the second law of thermodynamics in its local form in order to constrain the transport coefficients and find that there is a unique one. We then take the non-relativistic limit and identify the dissipative terms related to the breaking of boost invariance. The results are applicable in general to fluids with an explicit breaking of the boost symmetry, not necessarily with Lifshitz scaling.

Since Lorentz or Galilean invariance are fundamental symmetries in any microscopic theory, for the case of

a strange metal we interpret its absence in the fluid as the effect of an external medium through which the fluid moves. Following this philosophy we study a simple generalization of Drude’s model, where rather than a weakly interacting gas the collective behaviour of the electrons is an almost perfect fluid. The effect of the medium is a drag term and the new dissipative term. A similar model for graphene was considered in [11], but to our knowledge the effects due to the breaking of boost invariance have never been considered before. We propose different measurements of the conductivity and the heat production that can be used to determine the new transport coefficient.

Interestingly, we also find that scaling arguments fix the resistivity to depend linearly on the temperature $\rho_{xx} \sim T$, which is the observed behaviour in experiments [12], under the reasonable assumption that the dependence on the density is linear. The result seems to be universal, in the sense that it is independent of the number of dimensions or the value of the dynamical exponent.

Hydrodynamics.- The generators of Lifshitz symmetry are time translation $P_0 = \partial_t$, spatial translations $P_i = \partial_i$, the scaling transformation $D = -zt\partial_t - x^i\partial_i$ and rotations. The subalgebra involving D , P_i and P_0 has commutation relations

$$[D, P_i] = P_i, \quad [D, P_0] = zP_0. \quad (2)$$

In a field theory the scaling symmetry is manifested as a Ward identity involving the components of the energy-momentum tensor

$$zT^0_0 + \delta^j_i T^i_j = 0. \quad (3)$$

At finite temperature $T^0_0 = -\varepsilon$, $T^i_j = p\delta^i_j$, leading to the equation of state

$$z\varepsilon = dp. \quad (4)$$

The Lifshitz algebra can be generalized for constant velocities u^μ , $u^\mu u_\mu = \eta_{\mu\nu} u^\mu u^\nu = -1$. We define the generators

$$P^\parallel = u^\mu \partial_\mu, \quad P^\perp_\mu = P^\nu_\mu \partial_\nu, \quad D = zx^\mu u_\mu P^\parallel - x^\mu P^\perp_\mu. \quad (5)$$

Where $P_\mu^\nu = \delta_\mu^\nu + u_\mu u^\nu$. Then, the momentum operators commute among themselves and

$$[D, P^\parallel] = zP^\parallel, \quad [D, P_\mu^\perp] = P_\mu^\perp. \quad (6)$$

The Ward identity associated to D becomes

$$zT_\nu^\mu u_\mu u^\nu - T_\nu^\mu P_\mu^\nu = 0. \quad (7)$$

It coincides with (3) only when $z = 1$, but leads to the equation of state (4) for any velocity. If one insists on preserving the Ward identity in its original form (3), then the resulting equation of state will be velocity dependent

$$z\varepsilon - dp + (z - 1)(\varepsilon + p)u_i^2 = 0. \quad (8)$$

We do not study this possibility, rather we take (7) as the correct form after a change of coordinates to a frame where the fluid is moving. Our choice is supported by the analysis of the model proposed in [13] as a gravitational dual to Lifshitz points, and we will detail the calculation elsewhere.

The conservation of the energy-momentum tensor determines the hydrodynamic equations $\partial_\mu T^{\mu\nu} = 0$. Lorentz symmetry forces the energy-momentum tensor to be symmetric, but in other cases, as in Lifshitz theories, this condition can be relaxed. This allows many new terms in the hydrodynamic energy-momentum tensor, but as usual there are ambiguities in the definition of the velocity and the constitutive relations. In order to fix them, we impose the Landau frame condition

$$T^{\mu\nu} u_\nu = -\varepsilon u^\mu. \quad (9)$$

Then, the generalized form of the energy-momentum tensor is

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu + p\eta^{\mu\nu} + \pi_S^{(\mu\nu)} + \pi_A^{[\mu\nu]} + (u^\mu \pi_A^{[\nu\sigma]} + u^\nu \pi_A^{[\mu\sigma]})u_\sigma. \quad (10)$$

The first line is the ideal part of the energy-momentum tensor, while in the second line we write the dissipative corrections. π_A is antisymmetric in its two indices and π_S is symmetric. In addition, the Landau frame condition imposes $\pi_S^{(\mu\nu)} u_\nu = 0$. In a theory with rotational invariance $\pi_A^{[ij]} = 0$. There may be other new transport coefficients in a theory with more conserved charges, however for the example of a single conserved global current we do not find any to first order in derivatives.

Assuming Lifshitz symmetry, the scaling dimensions of temperature, energy, pressure and velocities are:

$$[T] = z, \quad [\varepsilon] = [p] = z + d, \quad [u^\mu] = 0. \quad (11)$$

Entropy current.- The new terms should be compatible with the laws of thermodynamics, in particular with the second law. Its local form in terms of the divergence of the entropy current is $\partial_\mu j_s^\mu \geq 0$. The divergence of

the entropy current can be derived from the conservation equation

$$0 = \partial_\mu T^{\mu\nu} u_\nu = -T \partial_\mu (s u^\mu) - \pi_A^{[\mu\sigma]} (\partial_{[\mu} u_{\sigma]} - u_{[\mu} u^\alpha \partial_\alpha u_{\sigma]}) + \dots \quad (12)$$

The dots denote contributions originating in symmetric terms in the energy-momentum tensor, to first order in derivative corrections they will simply be the shear and bulk viscosity contributions, which are manifestly positive for positive values of the transport coefficients. In order for the antisymmetric contribution to be positive, we should be able to write it as a sum of squares. This is possible only if

$$\pi_A^{[\mu\sigma]} = \alpha(\mu\sigma)(\partial_{[\mu} u_{\sigma]} - u_{[\mu} u^\alpha \partial_\alpha u_{\sigma]}), \quad (13)$$

where $\alpha(\mu\nu) = \alpha(\nu\mu)$, $\alpha(\mu\mu) = 0$ are new transport coefficients. The second law demands

$$\alpha(0i) \geq 0, \quad \alpha(ij) \leq 0. \quad (14)$$

The coefficients $\alpha(0i)$ break boost invariance and $\alpha(ij)$ rotational invariance. If only boost invariance is broken, then $\alpha(0i) \equiv \alpha$, $\forall i$. For a theory with Lifshitz symmetry the scaling dimension is $[\alpha] = d$, which determines the temperature dependence of the new transport coefficient to be

$$\alpha \sim T^{\frac{d}{z}}. \quad (15)$$

Kubo formula for isotropic fluids.- Under the transformation $x^i \rightarrow x^i + \xi^i$, the expectation value of the energy-momentum tensor is shifted, to leading order, by

$$\langle T^{\mu\nu}(x) \rangle = \int d^{d+1}x' \langle T^{\mu\nu}(x) T^{\alpha j}(x') \rangle \partial_\alpha \xi_j(x'). \quad (16)$$

The time derivative $v^i = \partial_0 \xi^i$ can be identified with a velocity. Then,

$$\pi_A^{[0i]} = \langle T^{[0i]} \rangle = \int d^{d+1}x' \langle T^{[0i]} T^{0j} \rangle v_j. \quad (17)$$

We have used that $\langle T^{[0i]} T^{jk} \rangle = 0$ by rotational invariance. Let us Fourier transform to frequency ω and momentum \vec{q} space the velocity and the correlation functions. Then, the first order transport coefficient is given by the Kubo formula

$$\alpha = - \lim_{\omega \rightarrow 0} \frac{1}{\omega} \frac{\delta_{ij}}{d} \text{Im} \langle T^{[0i]} T^{0j} \rangle (\omega, \vec{q} = 0). \quad (18)$$

Non-relativistic limit.- The relativistic velocity is $u^\mu = (1, \beta^i)/\sqrt{1 - \beta^2}$, where $\beta^i = v^i/c$. In the non-relativistic limit $\beta_i \rightarrow 0$, the pressure is not affected while the relativistic energy is expanded in terms of the mass density ρ and the internal energy U as

$$\varepsilon = c^2 \rho - \frac{\rho v^2}{2} + U. \quad (19)$$

The relativistic hydrodynamic equations reduce to the non-relativistic form

$$\partial_t \rho + \partial_i (\rho v^i) = 0, \quad (20)$$

$$\begin{aligned} \partial_t U + \partial_i (U v^i) + p \partial_i v^i \\ = \frac{\eta}{2} \sigma^{ij} \sigma_{ij} + \frac{\zeta}{d} (\partial_i v^i)^2 + \frac{\alpha}{2} (V_A^i)^2, \end{aligned} \quad (21)$$

$$\begin{aligned} \partial_t (\rho v^i) + \partial_j (\rho v^j v^i) + \partial^i p \\ = \partial_j \left(\eta \sigma^{ij} + \frac{\zeta}{d} \delta^{ij} \partial_k v^k \right) \\ + \partial_t (\alpha V_A^i) + \partial_j \left(\frac{\alpha}{2} (v^j V_A^i + v^i V_A^j) \right). \end{aligned} \quad (22)$$

The shear tensor is $\sigma_{ij} = \partial_i v_j + \partial_j v_i - (2/d) \delta_{ij} \partial_k v^k$. While taking the limit, we have absorbed factors of $1/c$ in the shear and bulk viscosities η and ζ and a factor $1/c^2$ in α . The vector V_A^i is defined as

$$V_A^i = 2D_t v^i + \omega^{ij} v_j, \quad (23)$$

where $D_t \equiv \partial_t + v^i \partial_i$ is the material derivative and $\omega_{ij} = 2\partial_{[i} v_{j]}$ is the tensor dual to the vorticity. The first term in V_A^i is proportional to the relative acceleration of the fluid, while the second term is proportional to an acceleration due to the Coriolis effect. Similarly to the viscosities, the coefficient α determines the dissipation that is produced in the fluid when the motion is not inertial.

The non-relativistic limit with $\rho \neq 0$ does not allow Lifshitz (or for that matter conformal) scaling in the original relativistic theory. However, one can have a non-relativistic version of Lifshitz scaling. Under a space-time diffeomorphism

$$t \rightarrow t + \xi^t, \quad x^i \rightarrow x^i + \xi^i, \quad (24)$$

the partition function of the theory will change as

$$\delta \log Z = \int dt d^d x \left(-\partial_\mu \xi^t j_\varepsilon^\mu + \partial_\mu \xi^i T_i^\mu \right). \quad (25)$$

The Lifshitz equation of state is recovered if the theory has a symmetry

$$\xi^t = zt, \quad \xi^i = x^i + \frac{z-2}{2} v^i t. \quad (26)$$

This is a combination of a scaling transformation (1) and a change of frame. The Ward identity becomes

$$0 = -z j_\varepsilon^t + \sum_i T_i^i + \frac{z-2}{2} v^i T_i^t = -zU + dp. \quad (27)$$

As for the relativistic case, imposing a scaling symmetry without the change of frame would modify the equation of state

$$-zU + dp = \frac{z-2}{2} \rho v^2. \quad (28)$$

Note, that in the relativistic case the analogous term in (4) that modifies the equation of state is proportional to $z-1$ rather than $z-2$. The reason is that, for $z=1$ in the relativistic case and $z=2$ in the non-relativistic, the Lifshitz symmetry group can be extended to the conformal groups that include boost transformations. The scaling Ward identity is then determined by a transformation that is independent of the velocity.

Conductivity.- Consider a charged relativistic fluid moving through a static medium with an external electromagnetic field applied to it. The hydrodynamic equations are

$$\partial_\mu J^\mu = 0, \quad \partial_\mu T^{\mu\nu} = F^{\mu\nu} J_\mu - \lambda c P_t^{\mu\nu} J_\nu, \quad (29)$$

where $P_t^{\mu\nu} = \eta^{\mu\nu} + t^\mu t^\nu$, $t^\mu = (1, 0)$, is the projector on the directions transverse to static medium. The coefficient λ determines the drag force that the fluid feels as it moves through the medium.

Let us define $E_i = F_{0i}$ and consider the case where the magnetic field is zero and $\partial_0 E_i = 0$. Projecting the equations of the energy-momentum tensor with t^μ and $P_t^{\mu\nu}$, we find

$$\partial_\mu T^{\mu 0} = J^i E_i, \quad \partial_\mu T^{\mu i} = J^0 E^i - \lambda c J^i. \quad (30)$$

We are interested in describing a steady state where the fluid has been accelerated by the electric field, increasing the current until the drag force is large enough to compensate for it. We assume that the flow does not change, but some scalar quantities like the energy can change with time. In order to simplify the calculation we will take the non-relativistic limit and consider only an incompressible fluid $\partial_i v^i = 0$. The fluid motion is described by the Navier-Stokes equations

$$\begin{aligned} \rho v^k \partial_k v^i + \partial^i p \\ = \rho E^i - \lambda \rho v^i + \eta \nabla^2 v^i + \frac{\alpha}{2} \partial_j ((v^j \sigma^{ik} + v^i \sigma^{jk}) v_k). \end{aligned} \quad (31)$$

We can solve this equation order by order in derivatives, keeping the pressure constant $\partial^i p = 0$. To leading order the current satisfies Ohm's law

$$J^i = \rho v^i \simeq \frac{\rho}{\lambda} E^i, \quad (32)$$

and the conductivity is simply $\sigma_{ij} = \rho/\lambda \delta_{ij}$. At higher orders in derivatives we find the following corrections for a divergenceless electric field $E_x(y)$

$$\sigma_{xx}(E_x) = \frac{\rho}{\lambda} \left[1 + \frac{1}{\rho \lambda E_x} \left(\eta \partial_y^2 E_x + \frac{\alpha}{6 \lambda^2} \partial_y^2 E_x^3 \right) \right]. \quad (33)$$

The conductivity depends on the electric field and its gradients. In the case where the electric field is linear $E_x = E_0 y/L$, the conductivity is simplified to

$$\sigma_{xx} = \frac{\rho}{\lambda} \left[1 + \frac{\alpha E_0^2}{\rho L^2 \lambda^3} \right]. \quad (34)$$

Note, that the contribution from the shear viscosity dropped. This gives a way to identify the new transport coefficient α , it can be measured experimentally as an enhancement of the conductivity with the electric field.

Another simple case is when the electric field takes the form $E_x = E_0 \cos(y/L)$. The contribution of α to the conductivity is y dependent

$$\sigma_{xx}(y) = \frac{\rho}{\lambda} \left[1 - \frac{\eta}{\lambda \rho L^2} + \frac{\alpha E_0^2}{\lambda^3 \rho L^2} - \frac{3\alpha E_x^2}{2\lambda^3 \rho L^2} \right]. \quad (35)$$

If we average on the y direction, we find again an enhancement of the conductivity with the electric field

$$\bar{\sigma}_{xx} = \frac{\rho}{\lambda} \left[1 - \frac{\eta}{\lambda \rho L^2} + \frac{\alpha E_0^2}{4\lambda^3 \rho L^2} \right]. \quad (36)$$

Lifshitz scaling.- In a fluid with Lifshitz symmetry the scaling dimensions of the hydrodynamic variables are

$$[v^i] = z - 1, \quad [p] = [U] = z + d, \quad [\rho] = d + 2 - z, \quad (37)$$

while the temperature has scaling dimension $[T] = z$. We can determine the scaling dimensions of the transport coefficients by imposing that all the terms in the hydrodynamic equations have the same scaling. We find

$$[\lambda] = z, \quad [\eta] = [\zeta] = d, \quad [\alpha] = d - 2(z - 1). \quad (38)$$

In contrast with a relativistic fluid, the density is approximately independent of the temperature. This introduces an additional scale, and in general the transport coefficients can be non-trivial functions of the ratio $\tau = T^{\frac{d+2-z}{z}}/\rho$. The conductivity will have the following temperature dependence

$$\sigma_{xx} = T^{\frac{d-2(z-1)}{z}} \hat{\sigma}(\tau) \simeq \frac{\rho}{T}, \quad (39)$$

where we assumed a linear dependence on the density as obtained from the calculation with the drag term. Note, that this predicts a resistivity linear in the temperature and *independent* of the dynamical exponent and the number of dimensions [12]. However, a prediction for the Hall angle that matches with the experiment $\cot \theta_H \sim T^2$ [14] is not as straightforward. The naïve scaling from this model would be $\cot \theta_H \sim T$. The scaling could be different if there is a strong temperature dependence of the permeability, while the permittivity is approximately constant.

Dissipative effects.- We can introduce constant homogeneous forces by electric fields or temperature gradients. They will induce an acceleration

$$a^i = -\partial^i p / \rho + E^i = (s/\rho) \partial^i T + E^i. \quad (40)$$

We assume $\partial_t a^i = 0$, $\partial_j a^i = 0$. The Navier-Stokes equations for homogeneous configurations takes the form

$$\partial_t v^i - 2(\alpha/\rho) \partial_t^2 v^i + \lambda v^i = a^i. \quad (41)$$

If we assume that the forces are suddenly switched on at $t = 0$, the evolution of the velocity is determined by this equation with the initial conditions $v^i(t = 0) = 0$,

$$\partial_t v^i(t = 0) = \frac{\rho a^i}{4\alpha\lambda} \left(\sqrt{\frac{8\alpha\lambda}{\rho} + 1} - 1 \right). \quad (42)$$

This choice is based on the physical requirement that at large times the velocity stays constant. When $\alpha \rightarrow 0$ it simply becomes $\partial_t v^i(t = 0) = a^i$.

The heat production rate induced by the force is

$$\partial_t U = \lambda \rho v^2 + 2\alpha(\partial_t v)^2. \quad (43)$$

At late times the system evolves to a steady state configuration with constant velocity, so the heat production rate becomes constant $v^i = a^i/\lambda$. Subtracting this contribution for all times, the total heat produced is

$$\Delta Q = -\frac{\rho a^2}{2\lambda^2} \left(\sqrt{\frac{8\alpha\lambda}{\rho} + 1} + 2 \right). \quad (44)$$

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- [1] P. Coleman, A. J. Schofield, *Nature (London)* **433**, 226 (2005).
 - [2] S. Sachdev, B. Keimer, *Phys.Today* **64N2**, 29 (2011).
 - [3] P. C. Hohenberg, B. I. Halperin, *Rev. Mod. Phys.* **49**, 435 (1977).
 - [4] R. M. Hornreich, M. Luban, S. Shtrikman, *Phys. Rev. Lett.* **35**, 1678 (1975).
 - [5] G. Grinstein, *Phys. Rev. B* **23**, 4615 (1981).
 - [6] P. Kovtun, D. Son, A. Starinets, *Phys.Rev.Lett.* **94**, 111601 (2005).
 - [7] C. Cao, *et al.*, *Science* **331**, 58 (2011).
 - [8] M. Müller, S. Sachdev, *Phys. Rev. B* **78**, 115419 (2008).
 - [9] L. Fritz, J. Schmalian, M. Müller, S. Sachdev, *Phys. Rev. B* **78**, 085416 (2008).
 - [10] M. Müller, J. Schmalian, L. Fritz, *Phys. Rev. Lett.* **103**, 025301 (2009).
 - [11] M. Mendoza, H. J. Herrmann, S. Succi, *Nature Scientific Reports* **3** (2013).
 - [12] M. Gurvitch, A. T. Fiory, *Phys. Rev. Lett.* **5**, 1337 (1987).
 - [13] S. Kachru, X. Liu, M. Mulligan, *Phys.Rev.* **D78**, 106005 (2008).
 - [14] T. R. Chien, Z. Z. Wang, N. P. Ong, *Phys. Rev. Lett.* **67**, 2088 (1991).